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Assignment Problems

If there are number of jobs (or tasks) which are to be assigned to an equal number of facilities (or people or machine) such that the assignment of a job to a particular person (or machine) would incur certain cost, then the problem of determining a set or sets of alternative assignment which would lead to an optimal solution "The best person for the job" is called an assignment problem.

The assignment problem involves finding the most efficient assignment of

- People to projects,
- Salespeople to territories,
- Contracts to bidders,
- Jobs to machines, etc.

The assignment model is actually a special case of the transportation model in which the workers represent the sources, and the jobs represent the destinations.

Let there be $\boldsymbol{m}$ jobs for $\boldsymbol{n}$ number of persons.
$\boldsymbol{c_{ij}}$ **= cost** of the assignment of the ***i-th*** job to ***j-th*** person.
We assume that a certain job can be assigned to certain person and conversely a certain person cannot do more than one job.
Decision variable is $\boldsymbol{x_{ij}} \in \{\mathbf{0}, \mathbf{1}\}$ which represents the assignment of **the *i-th*** job to ***j-th*** person.

The objective is to minimize the total cost.

$$Minimize\ total\ cost\ z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Where $x_{ij} = \begin{cases} 1 & if\ the\ i^{th}\ job\ is\ assigned\ to\ the\ j^{th}\ person \\ 0 & if\ the\ i^{th}\ job\ is\ not\ assigned\ to\ the\ j^{th}\ person \end{cases}$
$i = 1,2,\dots,m; \quad j = 1,2,\dots,n$

We have

$$\sum_{i=1}^{m} x_{ij} = 1, \quad j = 1,2,\dots,n$$

For a fixed j, only one $x_{ij} = 1$ for certain value i (only one job can be assigned to a person), and rest of $(n-1)x_{ij}$ are zeros. Similarly, since for fixed i, a particular job can only assign to one person, we must have

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,2,\dots,m$$

We can write the assignment problem as

$$\boldsymbol{Min} \quad \boldsymbol{z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}}$$

$$\boldsymbol{Subject\ to} \sum_{i=1}^{m} \boldsymbol{x_{ij} = 1},$$

$$\sum_{j=1}^{n} \boldsymbol{x_{ij} = 1},$$

$$\boldsymbol{x_{ij}} = \begin{cases} \boldsymbol{1 \ \ if\ the\ i^{th}\ job\ is\ assigned\ to\ the\ j^{th}\ person} \\ \boldsymbol{0 \ \ if\ the\ i^{th}\ job\ is\ not\ assigned\ to\ the\ j^{th}\ person} \end{cases}$$

$$\boldsymbol{i = 1,2,\dots,m; \quad j = 1,2,\dots,n}, \quad \boldsymbol{x_{ij}} \in \{\boldsymbol{0,1}\}$$

1- If $n = m$, then the problem is balanced.
2- If $n \neq m$, then the problem is unbalanced.

The assignment problem can also be written in the form of a $n \times n$ matrix called the cost of effectiveness matrix $[\boldsymbol{c_{ij}}]$.

		Workers (persons)				
		W_1	W_2	W_3	.....	W_n
Jobs (Tasks)	J_1	C_{11}	C_{12}	C_{13}	.....	C_{1n}
	J_2	C_{21}	C_{22}	C_{23}	.....	C_{2n}
	J_3	C_{31}	C_{32}	C_{33}	.....	C_{3n}
	:	:	:	:		:
	J_n	C_{n1}	C_{n2}	C_{n3}	.....	C_{nn}

Solving the assignment problem by the Hungarian Method

Although the assignment problem can be solved as an ordinary transportation problem or as a linear programming problem, its special structure can be exploited, resulting in a special-purpose algorithm, the so-called Hungarian Method.

It was developed and published in 1955 by Harold Kuhn, who gave the name "Hungarian method" because the algorithm was largely based on the earlier works of two Hungarian mathematicians: Dénes Kőnig and Jenő Egerváry.

Hungarian Method Procedure

Step 1: Subtract the smallest element of each row in the cost matrix from that row. Then subtract the smallest element of each column from that column. These steps create zeros.

Step 2: Make assignment in terms of zeros. This can be done as follows:

i- Starting with first row, if the row contains exactly one zero, mark ⊡ this zero, **as assignment will be made**.

ii- Draw a vertical line though that column containing this assigned zero.

iii- After examining all the rows, do the same to the columns. Starting from first column, examine all uncrossed columns to find a column has exactly one zero and mark ⊡ this zero and draw a horizontal line.

We will get two cases

a- All zeros have been marked,

b- The remaining zeros are at least two in each row and column.

If the number of marked zeros is **exactly** *n*, then we obtain the optimal solution.
If the number of marked zeros is **less than** *n*, then we shall modify the cost matrix.
Go to Step 3.

Step 3: Select the smallest element in the matrix, which do not lie on any of the lines and subtract this element from all uncrossed elements and add it to the elements which are placed at the intersection of horizontal and vertical lines.
Now a new matrix is formed and then we apply Step 2 to it until minimum number of lines becomes equal.

Step 4: Exactly one marked zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked zeroes will give the optimal assignment.

Notes:

- If number of lines less than *n*, then the assignment is not complete, if it is exactly *n*, then we get the optimal solution.
- No line passes through more than one marked zero.
- When we examine the resulting matrix from step1, if none of the rows or columns contain exactly one zero, then we start searching for two zeros.

Example1: Four jobs are 1, 2, 3, 4 to be assigned to four machines I, II, III, IV, the processing costs are as given in the matrix shown below. Find the assignment which will minimize the overall processing cost.

Machine / job	I	II	III	IV
1	12	30	21	15
2	18	33	9	31
3	44	25	24	21
4	23	30	28	14

Solution:
Number of Jobs = Number of Machines
Step 1: Subtract smallest element from each row, the Subtract smallest element from each column

Machine / job	I	II	III	IV
1	0	18	9	3
2	9	24	0	22
3	23	4	3	0
4	9	16	14	0

Machine / job	I	II	III	IV
1	0	14	9	3
2	9	20	0	22
3	23	0	3	0
4	9	12	14	0

Now we search rows for only one zero (if not then two zeros)

1- 1th row select (1, 1) and mark it, then draw a vertical line.
2- 2th row select (2, 3) and mark it, then draw a vertical line.
3- 4th row select (4, 4) and mark it, then draw a vertical line.

Now examine the columns.

1- We see that 2nd column is uncrossed at (3, 2), we assign the zero and a horizontal line.

Machine / job	I	II	III	IV
1		14	9	3
2	9	20		22
3	23		3	0
4	9	12	14	

The optimal assignment to minimize the overall processing cost:

Job: 1 2 3 4
Machine: I III II IV

Total cost is Z= $\sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}x_{ij}$ $\Rightarrow$ Z = 12 + 9 + 25 + 14 = 50

Example2: Solve the following minimal assignment problem.

job \ Man	I	II	III	IV
1	2	3	4	5
2	4	5	6	7
3	7	8	9	8
4	3	5	8	4

Solution:
Number of Jobs = Number of Men
Step 1: Subtract smallest element from each row, the Subtract smallest element from each column

job \ Man	I	II	III	IV
1	0	1	2	3
2	0	1	2	3
3	0	1	2	1
4	0	2	5	1

job \ Man	I	II	III	IV
1	0	0	0	2
2	0	0	0	2
3	0	0	0	0
4	0	1	3	0

None of the rows or columns contain exactly one zero so we search for two zeros.

1- 4th row select (4, 1) and mark it, then draw a vertical line.
2- 1st row select (1, 2) or (1, 3), we choose (1, 2) mark it, and then draw a vertical line.
3- 2nd row select (2, 3) and mark it, then draw a vertical line.
4- 3rd row select (3, 4), we mark it and draw a vertical line.

job \ Man	I	II	III	IV
1	**0**		**0**	**2**
2	**0**	**0**		**2**
3	**0**	**0**	**0**	
4		**1**	**3**	**0**

The optimal assignment to minimize the overall processing cost:

Job: 1 2 3 4

Man: II III IV I

Total cost is $Z=\sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}x_{ij} \Rightarrow Z = 3 + 6 + 8 + 3 = 20$

Note: Another optimal solution is (4, 1) + (1, 3) + (2, 2) + (3, 4) = 20.

Note: In the video I regard men as machines (it is the same not get confuse)

Example3: Solve the following minimal assignment problems.

job \ Man	I	II	III	IV	V
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

Solution:

Number of Jobs = Number of Men

Step 1: Subtract smallest element from each row, the Subtract smallest element from each column

job \ Man	I	II	III	IV	V
1	3	9	0	8	12
2	3	1	6	0	9
3	1	4	3	0	4
4	4	7	0	11	9
5	4	0	2	1	5

job \ Man	I	II	III	IV	V
1	2	9	0	8	8
2	2	1	6	0	5
3	0	4	3	0	0
4	3	7	0	11	5
5	3	0	2	1	1

We choose row1 at (1,3), row2 at (2,4), then row5 at (5,2) and draw vertical lines

job \ Man	I	II	III	IV	V
1	**2**	**9**	▢	**8**	**8**
2	**2**	**1**	**6**	▢	**5**
3	▢	**4** التقاطع	**3** تحت	**0** نقاط	**0**
4	3	7	0	11	5
5	**3**	▢	**2**	**1**	**1** اصغر رقم

We can see row 4 and column 5 have got no assignment. We apply Step3 choose the smallest of the elements that do not have a line through them is **1.** Subtract 1 from all elements that do not have a line through them and add it to elements at the intersection of two lines. We do a new assignment again.

Man / job	I	II	III	IV	V
1	**1**	**9**		**8**	**7**
2	**1**	**1**	**6**		**4**
3		**5**	**4**	**1**	**0**
4	2	7	0	11	4
5	**2**		**2**	**1**	**0**

We have assignment at row1at (1, 3), then row 2 at (2, 4), I draw vertical lines, then we search columns and assign column 1 at (3, 1), column 2 at (5, 2), and draw horizontal lines. Still row 4 and column 5 has no assignment so we repeat Step 2. Still we need to apply Step3 and choose smallest element 1.

Man / job	I	II	III	IV	V
1		**8**	**0**	**8**	**6**
2	**0**	**0**	**6**		**3**
3	**0**	**5**	**5**	**2**	
4	1	6		11	3
5	**2**		**3**	**2**	**0**

We have assignment to all rows. We search for only one zero and choose (4, 3), then (1, 1), (3, 5), (5, 2) with vertical zeros and (2, 4) with horizontal line.
The optimal assignment to minimize the overall processing cost:
Job: 1 2 3 4 5
Man:

$$Min \quad z = \sum_{i=1}^{5}\sum_{j=1}^{5} c_{ij}x_{ij} =$$

Unbalanced Assignment Problem

If the cost matrix of an assignment problem is not a square matrix, then the assignment problem is called an unbalanced assignment problem. In this case, dummy rows and columns are added in the matrix so as to form a square matrix. Then the usual assignment algorithm can be applied to this resulting balanced problem as shown in the following examples:

A) Dummy row

		Machines			
		M_1	M_2	M_3	M_4
Jobs	J_1	20	25	22	28
	J_2	15	18	23	17
	J_3	19	17	21	24
	(Dummy)	0	0	0	0

B) Dummy Column

		Machines			
		M_1	M_2	M_3	(Dummy) M_4
Jobs	J_1	20	25	22	0
	J_2	15	18	23	0
	J_3	19	17	21	0
	J_4	19	17	21	0

Example: A department head has four tasks to be performed and three subordinates. The subordinates differ in efficiency. The estimates of the time, each subordinates would take to perform is given below in the matrix. How should be allocate the tasks, one to each man, so as to minimize the total man hour?

Tasks \ Men	1	2	3
I	9	26	15
II	13	27	6
III	35	20	15
IV	18	30	20

Solution:

Number of Jobs ≠ Number of Men ⇒ $4 \neq 3$

Dummy column is added in the matrix:

Tasks \ Men	1	2	3	4
I	9	26	15	0
II	13	27	6	0
III	35	20	15	0
IV	18	30	20	0

Number of Jobs = Number of Men $\Rightarrow$ 4 = 4
Step 1: Subtract smallest element from each row, then Subtract smallest element from each column

Men / Tasks	1	2	3	4
I	0	6	9	0
II	4	7	0	0
III	26	0	9	0
IV	9	10	14	0

Now we search rows for only one zero (if not then two zeros)

1- 4th row select (4, 4) and mark it, then draw a vertical line.
   Now examine the columns.
2- 1st column select (1, 1) and mark it, then draw a horizontal line.
3- 2th column select (3, 2) and mark it, then draw a horizontal line.
4- 2nd column select (2, 3) and mark it, then draw a horizontal line.

Men / Tasks	1	2	3	4
I	0	6	9	0
II	4	7	0	0
III	26	0	9	0
IV	9	10	14	0

Job I $\rightarrow$ Man1 =
Job II $\rightarrow$ Man 3 =
Job III $\rightarrow$ Man 2 =
Job IV $\rightarrow$ Man 4 =

Total cost is Z= $\sum_{i=1}^{4}\sum_{j=1}^{4} c_{ij}x_{ij}$ $\Rightarrow$ Z =

H.W: Four jobs A B C D are to be assigned to four machines X Y L Z. The processing costs are as given in the matrix shown below. Find the assignment which will minimize the overall processing cost.

	A	B	C	D
X	25	22	30	34
Y	20	23	28	20
L	27	25	25	30
Z	18	24	21	25

H.W: The matrix below gives the cost in dollars of producing job on machine. How the jobs should be assigned to the various machines so that the total cost is minimized.

		Machines		
		M_1	**M_2**	**M_3**
Jobs	J_1	9	12	11
	J_2	8	13	17
	J_3	20	12	13
	J_4	21	15	17

H.W: Four persons A, B, C and D are to be assigned four jobs I, II, III and IV. The cost matrix is given as under, find the proper assignment.

Man / Jobs	*A*	*B*	*C*	*D*
I	8	10	17	9
II	3	8	5	6
III	10	12	11	9
IV	6	13	9	7

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